Lesson 6-4 Fund Thm Calculus Part 1 Key

AP Calculus AB
Lesson 6-4: *The Fundamental Theorem of Calculus, Part 1*

Name Hem/ 2018
Date

Learning Goals:

- *I can apply the Fundamental Theorem of Calculus.*
- *I understand the relationship between the derivative and definite integral as expressed in both parts of the Fundamental Theorem of Calculus.*

As discussed earlier, once the developers of Calculus (Newton and Leibniz) had the method for finding how functions change at a given instant (*differential calculus*), they needed a method to describe how those instantaneous changes could accumulate over an interval to produce the original function. This reason is why they also investigated *areas under curves*, which ultimately led to the second main branch of calculus, called *integral* calculus. Once Newton and Leibniz had the calculus for finding slopes of tangent lines and the calculus for finding areas under curves, two geometric operations that would seem to have nothing at all to do with each other, the challenge for them was to prove the connection. The discovery of this connection (The Fundamental Theorem of Calculus) is probably the single most powerful discovery in the history of mathematics. Now, let's work our way towards this discovery.

Antiderivatives Reviewed

Remember back from Lesson 5-2 (when we learned the *Mean Value Theorem*) that we learned about antiderivative. A function $F(x)$ is an **antiderivative** of a function $f(x)$ if $F'(x)=f(x)$ for all x in the domain of *f*.

Practice 1

1. The function $f(x)=4x+5$. The function $F(x)$, the antiderivative of $f(x)$, contains the coordinate $(-1,7)$. Find $F(x)$.

$F(x)=\frac{4}{2}x^2+5x+c$

$F(x)=2x^2+5x+C$

$7=2(-1)^2+5(-1)+C$

$7=2-5+C$

$10=C$

$\boxed{F(x)=2x^2+5x+10}$

2. The function $g(x)=\cos\left(\frac{1}{2}x\right)$. The function $G(x)$, the antiderivative of $g(x)$, contains the coordinate $\left(\frac{\pi}{2},1+\sqrt{2}\right)$. Find $G(x)$.

$G(x)=2\sin\left(\frac{1}{2}x\right)+C$

$1+\sqrt{2}=2\sin\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)+C$

$1+\sqrt{2}=2\sin\frac{\pi}{4}+C$

$1+\sqrt{2}=\cancel{2}\cdot\frac{\sqrt{2}}{\cancel{2}}+C$

$1+\cancel{\sqrt{2}}=\cancel{\sqrt{2}}+C$

$C=1$

$\boxed{G(x)=2\sin\left(\frac{1}{2}x\right)+1}$

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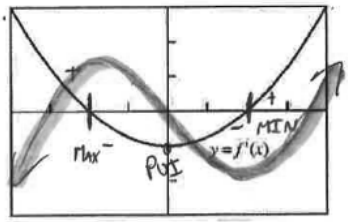
Graphical Antidifferentiation

Each of the following graphs represents the derivative of a continuous function f. Sketch a possible graph of $y = f(x)$ on the same set of axes as the derivative, assuming $f(0) = 0$



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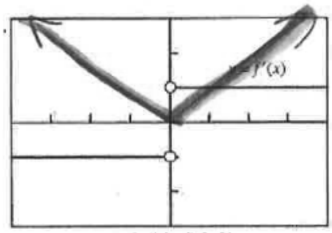

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The Relationship between Antiderivatives and Integrals – **Finding the Derivative of an Integral**

The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a,b]$, then the function

$$F(x) = \int_a^x f(t)\,dt$$

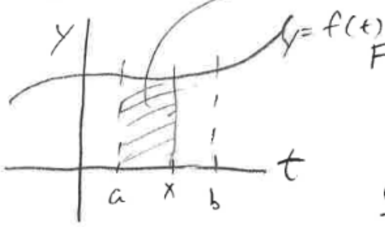
has a derivative at every point x in $[a,b]$, and

$$\boxed{\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

This astonishing connection between differentiation and integration, in the words of the authors of our textbook, "fueled the scientific revolution for the next 200 years, and is still regarded as the most important computational discovery in the history of mathematics." Due to this importance, the discovery is called the **Fundamental Theorem of Calculus**.

★ *Part 1 of the Fundamental Theorem of Calculus says that the definite integral of a continuous function is a differentiable function of its upper limit of integration.*

Notes from Video: **https://www.youtube.com/watch?v=C7ducZoLKgw**



$F(x) = \int_a^x f(t)\,dt$, where x is in $[a,b]$

According to FTC

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$\therefore \frac{dF}{dx} = f(x)$$

Lower Boundary does not matter!

★ Every cont. function has an antiderivative

★ Connects Differential Calc. & Integral Calc.

Why is the FTC the Bombdiggity?

1. **Every continuous function has an antiderivative.**

2. **Tells us the processes of differentiation and integration are inverses of each other.**

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Example 1

a. Find $\frac{d}{dx}\int_{-\pi}^{x}\cos t\,dt$.

Lower bound # does not matter!

$= \boxed{\cos x}$

b. Find $\frac{d}{dx}\int_{0}^{x}\frac{1}{1+u^2}\,du$.

$= \boxed{\frac{1}{1+x^2}}$

Example 2

Find $\frac{dy}{dx}$ if $y=\int_{1}^{x^2}\cos t\,dt$.

A chain rule! Recall: $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$

Let $u=x^2$

$\frac{du}{dx}=2x$

$\frac{dy}{dx}=\frac{d}{dx}\int_{1}^{u}\cos t\,dt\cdot\frac{du}{dx}$

$=\cos u\cdot 2x$

$=\cos x^2\cdot 2x = \boxed{2x\cos x^2}$

Practice 2

1. Find $\frac{dy}{dx}$ if $y=\int_{-\pi}^{x}\sin^2 t\,dt$.

$\frac{dy}{dx}=\boxed{\sin^2 x}$

2. Find $\frac{dy}{dx}$ if $y=\int_{0}^{x}\left(3t+\cos t^2\right)dt$.

$\frac{dy}{dx}=\boxed{3x+\cos x^2}$

3. Find $\frac{dy}{dx}$ if $y=\int_{0}^{x^2}e^{t^2}\,dt$.

Chain rule

Let $u=x^2$

$\frac{du}{dx}=2x$

$\frac{dy}{dx}=\frac{d}{du}\int_{0}^{u}e^{t^2}dt\cdot\frac{du}{dx}$

$=e^{u^2}\cdot 2x$

$=e^{(x^2)^2}\cdot 2x=\boxed{2xe^{x^4}}$

4. Find $\frac{dy}{dx}$ if $y=\int_{\pi}^{\pi-x}\frac{1+\sin^2 t}{1+\cos^2 t}\,dt$.

Chain rule

Let $u=\pi-x$

$\frac{du}{dx}=-1$

$\frac{dy}{dx}=\frac{d}{du}\int_{\pi}^{u}\frac{1+\sin^2 t}{1+\cos^2 t}\cdot\frac{du}{dx}$

$=\frac{1+\sin^2 u}{1+\cos^2 u}\cdot -1$

$=\boxed{-\frac{1+\sin^2(\pi-x)}{1+\cos^2(\pi-x)}}$

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Example 3

We will need to use your rules of integration to rewrite these before using the FTC, Part 1.

a. Find $\frac{dy}{dx}$ if $y=\int_x^8 \sqrt{t^2+6t-4}\,dt$.

$$y=-\int_8^x \sqrt{t^2+6t-4}\,dt$$

$$\frac{dy}{dx}=\boxed{-\sqrt{x^2+6x-4}}$$

b. Find $\frac{dy}{dx}$ if $y=\int_{2x}^{x^2} \frac{1}{2+e^t}\,dt$.

$$y=\int_0^{x^2}\frac{1}{2+e^t}\,dt-\int_0^{2x}\frac{1}{2+e^t}\,dt$$

Let $u=x^2$, $\frac{du}{dx}=2x$

Let $v=2x$, $\frac{dv}{dx}=2$

$$\frac{dy}{dx}=\frac{d}{du}\int_0^u\frac{1}{2+e^t}\,dt\cdot\frac{du}{dx}-\frac{d}{dv}\int_0^v\frac{1}{2+e^t}\cdot\frac{dv}{dx}$$

$$=\frac{1}{2+e^u}\cdot 2x-\frac{1}{2+e^v}\cdot 2$$

$$=\boxed{\frac{2x}{2+e^{x^2}}-\frac{2}{2+e^{2x}}}$$

Example 4

Find a function $y=f(x)$ with derivative $\frac{dy}{dx}=\tan x$ that satisfies the condition $f(3)=5$.

Lower limit

"C"

$$y=\int_3^x \tan t\,dt$$

$$\boxed{f(x)=\int_3^x \tan t\,dt+5}$$

Practice 3

Construct a function that has a derivative of $\frac{dy}{dx}=\sin^3 x$, and $y=0$ when $x=5$. $f(5)=0$

$$y=\int_5^x \sin^3 t\,dt$$

$$\boxed{f(x)=\int_5^x \sin^3 t\,dt+0}$$

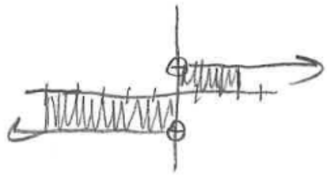
Mixed Review

2016 AP Exam ***NO CALCULATOR***

$$f(x) = \begin{cases} \dfrac{|x|}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

9. The function f is defined above. The value of $\int_{-5}^{3} f(x)\,dx$ is

$\boxed{\text{(A) } -2}$ (B) 2 (C) 8 (D) nonexistent



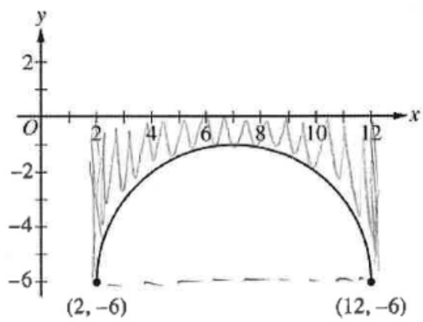
$A = -5 + 3 = -2$

11. The graph of $y = f(x)$ consists of a semicircle with endpoints at $(2, -6)$ and $(12, -6)$.

What is the value of $\int_{2}^{12} f(x)\,dx$?

(A) $-\frac{25\pi}{2}$ (B) $\frac{25\pi}{2}$

(C) $-60 + \frac{25\pi}{2}$ (D) $60 - \frac{25\pi}{2}$



Graph of f

$A = -\left[60 - \frac{\pi(25)}{2}\right] = -60 + \frac{25\pi}{2}$

under the x-axis

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1. Given the equation $\int_1^5 f(x)dx = 10$ and $\int_1^4 f(x)dx = 6$.

Find: $2\int_4^5 f(x)dx$

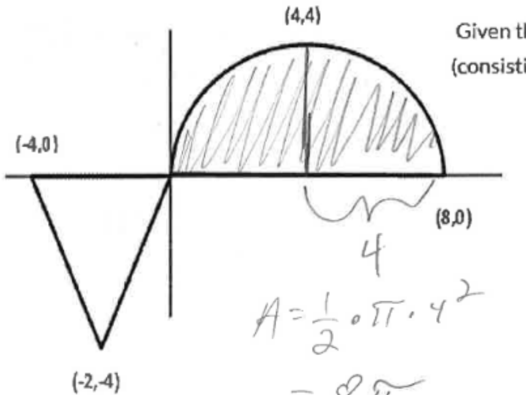
$\boxed{\text{A} \quad -8}$ C 4

B -4 D 8

$2\left[-\int_5^1 f(x)\,dx + \int_1^4 f(x)\,dx\right] =$

$2\left[-1 \cdot 10 + 6\right] =$

$2 \cdot -4 = -8$

2. 

4

$A = \frac{1}{2} \cdot \pi \cdot 4^2$

$= 8\pi$

Given the graph of $f(x)$ (consisting of a triangle and semicircle), what is the $\int_0^8 f(x)dx$?

A 8π C 32π

B 16π D 64π

3. Use a left Riemann Sum with 4 equal subintervals to approximate:

$\Delta x = \frac{5-1}{4} = \frac{4}{4} = 1$

$$\int_1^5 \frac{1}{x}dx$$

A $\ln 5$ C $\frac{77}{60}$

$\boxed{\text{B} \quad \frac{25}{12}}$ D $\frac{101}{60}$

1	2	3	4	5
1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$

$\text{Area} = 1\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)$

$= 1\left(\frac{12}{12} + \frac{6}{12} + \frac{4}{12} + \frac{3}{12}\right) = \frac{25}{12}$